FORMALITY OF THE CONSTRUCTIBLE DERIVED CATEGORY FOR SPHERES: A COMBINATORIAL AND A GEOMETRIC APPROACH

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ABSTRACT. We describe the constructible derived category of sheaves on the n-sphere, stratified in a point and its complement, as a dg module category of a formal dg algebra. We prove formality by exploring two different methods: As a combinatorial approach, we reformulate the problem in terms of representations of quivers and prove formality for the 2-sphere, for coefficients in a principal ideal domain. We give a suitable generalization of this formality result for the 2-sphere stratified in several points and their complement. As a geometric approach, we give a description of the underlying dg algebra in terms of differential forms, which allows us to prove formality for n-spheres, for real or complex coefficients.

1. INTRODUCTION

Let X be a stratified topological space. A sheaf of modules over a principal ideal domain R on X is called constructible if it is locally constant along the strata, and the stalks are finitely generated. Let $D^b(X)$ be the bounded derived category of sheaves of R-modules on X. The constructible derived category $D^b_c(X)$ is the full triangulated subcategory of $D^b(X)$ consisting of bounded complexes of sheaves with constructible cohomology.

In this paper we give an algebraic description of $D^b_c(X)$ in terms of dg module categories. We would like to prove for the special case of the n-spheres that the corresponding dg algebra is formal, i.e. quasi-equivalent to a dg algbra with trivial differential. For this purpose, we give general combinatorial and geometric descriptions of $D^b_c(X)$, which we then use to prove the desired formality results.

The combinatorial approach relies on a generalization of a result of [KS94], who showed that for a simplicial complex $\mathcal{K}$, we have an equivalence of categories

$$D^b_c(\mathcal{K}) \simeq D^b(\mathrm{Sh}_c(\mathcal{K}))$$

where $\mathrm{Sh}_c(\mathcal{K})$ denotes the category of constructible sheaves on $\mathcal{K}$ (with respect to the natural stratification). We prove a similar statement for more general stratifications, which we call acyclic. As a consequence, we get the following combinatorial description of $D^b_c(X)$:

Theorem 1. *(cf. Theorem 21) To every acyclic stratification on a topological space X of finite homological dimension we can assign a canonical quiver Q with relations ρ, such that we get a natural equivalence*

$$D^b_c(X) \xrightarrow{\sim} D^b(\mathrm{Rep}_{\mathrm{f}}(Q,\rho))$$

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where $\mathrm{Rep_f}(Q,\rho)$ *denotes the category of representations of* (Q,ρ) *with finitely generated stalks.*

For the geometric approach, we give a description of $D^b_c(X)$ in terms of differential forms, for the case that X is a differentiable manifold, stratified in a point and its complement, and the base ring is the field $\mathbb{K}$ of real or complex numbers. Inspired by results of [DGMS75], who proved that the de Rham algebra Ω on a Kähler manifold is formal, we define a suitable "extended" de Rham algebra $\mathcal{M}(\Omega)$, for which we get the following characterization of $D^b_c(X)$:

Theorem 2. *(cf. Theorem 13) Let* X *be a second countable differentiable manifold and* pt *a point in* X *such that* $X\backslash pt$ *is simply connected. We have an equivalence of triangulated categories*

$$D^b_c(X) \simeq \mathcal{D}^f_{\mathcal{M}(\Omega)}$$

where $\mathcal{D}^f_{\mathcal{M}(\Omega)}$ *is a suitable subcategory of the derived dg module category* $(\mathcal{D}_{\mathcal{M}(\Omega)})^\circ$.

We use those two descriptions of $D^b_c(X)$ to prove formality for the n-sphere S^n, stratified in a point and its complement. For this, let $\mathcal{L}_1$ be the skyscraper at the point of the stratification, and $\mathcal{L}_2 = R_{S^n}[\lfloor n/2 \rfloor]$ the constant sheaf on the n-sphere shifted by $\lfloor n/2 \rfloor$. For $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$, consider the dg algebra $\mathrm{Ext}(\mathcal{L})$ of self-extensions of $\mathcal{L}$ in $D^b(S^n)$, with trivial differential. Let $\mathcal{D}_{\mathrm{Ext}(\mathcal{L})}$ be the derived category of dg modules over $\mathrm{Ext}(\mathcal{L})$ as defined in [BL94], and $\mathcal{D}^f_{\mathrm{Ext}(\mathcal{L})}$ the full triangulated subcategory of $(\mathcal{D}_{\mathrm{Ext}(\mathcal{L})})^\circ$ generated by the dg modules $\mathcal{L}p_1$ and $\mathcal{L}p_2$, for $p_i : \mathcal{L} \to \mathcal{L}_i$ the projections. We prove the following formality results:

Theorem 3. (cf. Theorems 14, 27)

(i) For R *any principal ideal domain, we have an equivalence*

$$D^b_c(S^2) \simeq \mathcal{D}^f_{\mathrm{Ext}(\mathcal{L})}$$

(ii) If the base ring R *is* $\mathbb{R}$ *or* $\mathbb{C}$*, we have for* $n \geq 2$ *an equivalence*

$$D^b_c(S^n) \simeq \mathcal{D}^f_{\mathrm{Ext}(\mathcal{L})}$$

The main step in the proof of this result is to take an injective resolution I of $\mathcal{L}$ and prove that $\mathrm{End}\, I$ is a formal dg algebra. This argument can be generalized to the following setting:

Theorem 4. (cf. Theorem 29) *For the 2-sphere* S^2 *stratified in* m *points and their complement, let* $\mathcal{L}_1, \ldots, \mathcal{L}_m$ *be the skyscrapers at the points of the stratification, and* $\mathcal{L} = R_{S^2}[1]$ *the constant sheaf shifted by 1. For any injective resolution* I *of* $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m \oplus \mathcal{L}$*, the corresponding dg algebra* $\mathrm{End}\, I$ *is formal.*

The idea of describing $D^b_c(S^n)$ as a dg module category of a formal dg algebra goes back to an analogous conjecture in the corresponding equivariant setting. For X a projective variety on which a complex reductive group G acts with finitely many orbits, let $D^b_{G,c}(X)$ be the equivariant constructible derived category as introduced in [BL94]. In [Soe01] it has been conjectured implicitely that $D^b_{G,c}(X)$ is equivalent to $\mathcal{D}^f_{\mathcal{E}}$, where $\mathcal{E}$ is the equivariant extension algebra on X, i.e. a dg algebra with trivial differential. So far, this conjecture has been proved in special cases ([Lun95],[Gui05]); a proof for flag varieties is due to appear in [Sch08].

Our above-mentioned results explore in how far the equivariant case can be translated to the non-equivariant setting. For the proofs of Theorems 3(i) and 4 we use the combinatorial description of $D^b_c(X)$ given in Theorem 1. This method works for coefficients in any principal ideal domain. For the proof of Theorem 3 (ii) we exploit Theorem 2, hence it only works for coefficients in $\mathbb{R}$ or $\mathbb{C}$.

Our paper is organized as follows: Section 2 is a summary of relevant results on the constructible derived category and dg module categories. In section 3, we describe the bounded constructible derived category $D^b_c(X)$, for X a differentiable manifold stratified in a point and its complement, using differential forms. This characterization allows us to prove formality for n-spheres with this specific stratification. In section 4 we introduce acyclic stratifications and give a combinatorial description of $D^b_c(X)$ in terms of representations of quivers. Using this description, we prove in section 5 formality for the 2-sphere, stratified in several points and their complement.

This paper is a condensed version of my diploma thesis [Bal06], which I wrote in Freiburg in 2005. I am deeply indebted to Wolfgang Soergel for his support and many fruitful discussions, and to Olaf Schnürer for lots of helpful comments.

2. The constructible derived category as a dg module category

2.1. **Conventions.** We fix a commutative unitary Noetherian ring R of finite homological dimension. By a sheaf on a topological space X we always mean a sheaf of modules over R. We denote the category of such sheaves by $\mathrm{Sh}(X)$, and by $\mathrm{Sh}_{\mathrm{f}}(X)$ the full subcategory of sheaves with finitely generated stalks. As usual, we write $D^b(X)$ for the bounded derived category of sheaves of R-modules on X.

We denote the constant sheaf with stalk R by R_X. For a continous map $f : X \to Y$ we write f^* for the inverse image functor, f_* for the direct image and $f_!$ for the direct image with compact support (if it exists). In this paper we are mostly going to use locally closed inclusions and would like to recall the following properties (cf. [Ive86]): If f is a locally closed inclusion then $f_!$ is exact and has a right adjoint $f^{(!)} : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, which is left exact and preserves injectives; its right derived $Rf^{(!)}$ yields the right adjoint of $f_!$ on the corresponding derived categories and is denoted by $f^!$, as usual. For open inclusions j we have $j^* = j^{(!)}$, and for closed inclusions i, $i_! = i_*$.

2.2. **Stratified spaces and the constructible derived category.** Let X be a topological space. By a stratification on X we mean a finite stratification such that

(i) the strata are simply connected topological manifolds;
(ii) they are locally closed in X; and
(iii) the closure of any stratum is again a union of strata.

The central objects of interest in this paper are constructible sheaves and the corresponding derived categories: A sheaf on a stratified topological space X is called weakly constructible if its restriction to every stratum is a constant sheaf, and constructible, if in addition all its stalks are finitely generated. The usual definition of constructibility is slightly different in that it requires a sheaf to be locally constant along the strata. However, since we only allow simply connected strata the two definitions coincide. We denote the corresponding categories of constructible and weakly constructible sheaves by $\mathrm{Sh}_c(X)$ and $\mathrm{Sh}_{w,c}(X)$, respectively.

A complex of sheaves is called (weakly) constructible if its cohomology sheaves are (weakly) constructible. For $* = +$ or b, we denote by $D^*_c(X)$ the full triangulated subcategories of $D^*(X)$ that consist of constructible complexes, and by $D^*_{w,\,c}(X)$ the corresponding weakly constructible category.

2.3. **The perfect category, quasi-equivalences and formality.** We assume the reader to be familiar with the concepts of dg agebra, dg module and the corresponding derived categories. Otherwise all that we need can be found in [BL94].

For a dg algebra $\mathcal{A}$, we denote by $\mathcal{M}_{\mathcal{A}}$ the category of (left) dg modules over $\mathcal{A}$, and by $\mathcal{D}_{\mathcal{A}}$ the corresponding derived dg module category. The perfect category $\mathrm{Per}_{\mathcal{A}}$ is the full triangulated subcategory of $\mathcal{D}_{\mathcal{A}}$ generated by the direct summands of $\mathcal{A}$.

A natural question is under which circumstances the perfect categories of two dg algebras are equivalent. The easiest case is that of two quasi-isomorphic dg algebras, for which [BL94] have shown that the corresponding perfect categories are equivalent. However, there is a more general result, for which we need the notion of quasi-equivalence. Two dg algebras $\mathcal{A}$ and $\mathcal{B}$ are called quasi-equivalent if there is an $\mathcal{A}$-$\mathcal{B}$-dg-bimodule M, together with a cycle of degree zero $c \in M$, such that the cohomology class of c is a basis of HM as $H\mathcal{A}$-left- and $H\mathcal{B}$-right-module. This means that the morphisms

$$\begin{aligned} \mathcal{A} &\to M, \quad a \mapsto a \cdot c \quad \text{and} \\ \mathcal{B} &\to M, \quad b \mapsto c \cdot b \end{aligned}$$

induce isomorphisms on the cohomology level. [Kel94] proved that for two quasi-equivalent dg algebras $\mathcal{A}$ and $\mathcal{B}$ the corresponding perfect categories $\mathrm{Per}_{\mathcal{A}}$ and $\mathrm{Per}_{\mathcal{B}}$ are equivalent.

This result leads us to the definition of formality. Usually a dg algebra is called formal if it is quasi-isomorphic (in a generalized sense) to its cohomology algebra, i.e. if we have a chain of quasi-isomorphisms

$$\mathcal{A} \to \mathcal{A}_1 \leftarrow \mathcal{A}_2 \to \cdots \leftarrow H\mathcal{A}$$

Considering the result just cited, we would like to introduce a more general notion of formality: We call a dg algebra $\mathcal{A}$ formal if it is (in a generalized sense) quasi-equivalent to its cohomology algebra, i.e. if we find dg algbras $\mathcal{A} = \mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_n = H\mathcal{A}$ such that $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ are quasi-equivalent. It follows that for a formal dg algebra $\mathcal{A}$ the categories $\mathrm{Per}_{\mathcal{A}}$ and $\mathrm{Per}_{H\mathcal{A}}$ will be equivalent.

2.4. **Derived categories as dg module categories.** We would now like to describe the constructible derived category by an equivalent dg module category. Before we get started we would like to remind the reader that for any complex $\mathcal{F}$ of objects in an R-linear abelian category, the corresponding endomorphism complex $\mathrm{End}(\mathcal{F})$ has the structure of a dg algebra in a natural way. Moreover, if we take two complexes $\mathcal{F}$ and $\mathcal{G}$ and denote by $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ the usual homomorphism complex (as can be found, for example, in [KS94]), $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$ will be a left dg module over $\mathrm{End}(\mathcal{G})$ and a right dg module over $\mathrm{End}(\mathcal{F})$. The following result, which is due to [Kel94], describes a general method for making the transition from subcategories of a derived category that are given by generators, to dg module categories.

Proposition 5. *Let $\mathcal{C}$ be an R-linear abelian category, $K^b(\mathcal{C})$ the category of bounded complexes over $\mathcal{C}$ and I a bounded complex of injectives in $\mathcal{C}$. Then the*

contravariant functor

$$\operatorname{Hom}(\cdot, I) : K^b(\mathcal{C}) \to \mathcal{M}_{\operatorname{End} I} \tag{6}$$

is fully faithful on the full triangulated subcategory of $D^b(\mathcal{C})$ that is generated by the direct summands of I.

Denote by $\langle I \rangle_\oplus$ the full triangulated subcategory of $D^b(\mathcal{C})$ that is generated by the direct summands of I. Following [Lun95], we denote by $\mathcal{D}^f_{\operatorname{End} I}$ the image of $\langle I \rangle_\oplus$ under the functor (6). Hence $\mathcal{D}^f_{\operatorname{End} I}$ is the subcategory of $(\operatorname{Per}_{\operatorname{End} I})^\circ$ generated by those direct summands of $\operatorname{End} I$ that arise from direct summands of I, i.e. by dg modules of the form $(\operatorname{End} I) \cdot p$, where p is a projector of I. Now Proposition 5 can be reformulated as follows:

Corollary 7. *The functor (6) induces an equivalence $\langle I \rangle_\oplus \simeq \mathcal{D}^f_{\operatorname{End} I}$.*

In order to apply this result to $D^b_c(X)$ we need a set of generators of this category: Denote for any locally closed subset $Z \subset X$ by R^X_Z the constant sheaf on Z extended by zero, i.e. the sheaf $i_! R_Z$, where $i : Z \hookrightarrow X$ is the inclusion. It is a well known fact that $D^b_c(X)$ is generated as a triangulated subcategory of $D^b(X)$ by its heart, i.e. by $\operatorname{Sh}_c(X)$. This can be shown using dévissage. As a consequence, if our base ring R is a principal ideal domain, $D^b_c(X)$ is generated by

$$\left\{ R^X_{\overline{S}} \, [d(S)] \mid S \text{ varies over the strata} \right\}$$

where $[d(S)]$ denotes the shift by $d(S) := \lfloor \frac{\dim S}{2} \rfloor$. If we take the direct sum of these generators and an injective resolution I of this direct sum, Corollary 7 yields an equivalence

$$D^b_c(X) \xrightarrow{\sim} \mathcal{D}^f_{\operatorname{End} I} \tag{8}$$

This is the central result that will allow us to describe the constructible derived category as a dg module category. However, injective resolutions are difficult to compute in general. We would therefore like to allow more general resolutions in Proposition 5, which leads us to the following result:

Proposition 9. *Let $\mathcal{C}$ be an R-linear abelian category with enough injectives. Let Ω be a bounded complex in $D^b(\mathcal{C})$ such that $\operatorname{Ext}^i(\Omega^p, \Omega^q) = 0$ for $i > 0$ and $p, q \in \mathbb{Z}$, and I an injective resolution of Ω that is bounded from below. Then $\operatorname{End} \Omega$ and $\operatorname{End} I$ are quasi-equivalent.*

Proof. Let $c : \Omega \to I$ be a quasi-isomorphism. It is a standard result of homological algebra that

$$\operatorname{End} I \to \operatorname{Hom}(\Omega, I), \quad a \mapsto a \circ c$$

is a quasi-isomorphism ([Ive86]). Now we only need to prove that

$$\operatorname{End} \Omega \to \operatorname{Hom}(\Omega, I), \quad b \mapsto c \circ b$$

is a quasi-isomorphism as well. This statement is a generalization of the well-known fact that, for a left-exact abelian functor F, its right derived RF can be calculated using F-acyclic objects ([KS94]). Using this result we get that for every $p \in \mathbb{Z}$ the map

$$\operatorname{Hom}(\Omega^p, \Omega) \to \operatorname{Hom}(\Omega^p, I)$$

induced by c is a quasi-isomorphism. Now consider the distinguished triangle

$$\Omega \xrightarrow{c} I \to M \xrightarrow{[1]},$$

where M is the mapping cone of c. Using the distinguished triangle

$$\mathrm{Hom}(\Omega^p,\Omega) \to \mathrm{Hom}(\Omega^p, I) \to \mathrm{Hom}(\Omega^p, M) \xrightarrow{[1]}$$

we see that $\mathrm{Hom}(\Omega^p, M)$ must have been an exact complex. If we can show that $\mathrm{Hom}(\Omega, M)$ is exact, we are done since we have another distinguished triangle

$$\mathrm{Hom}(\Omega,\Omega) \to \mathrm{Hom}(\Omega, I) \to \mathrm{Hom}(\Omega, M) \xrightarrow{[1]}$$

But $\mathrm{Hom}(\Omega, M)$ is the product total complex of the double complex $\mathrm{Hom}(\Omega^p, M^q)$, which is bounded from left and below and whose columns are exact, as seen above. For this kind of double complexes it has been shown in [KS94] that their product total complex is exact. □

3. Formality for simple stratifications on manifolds

3.1. **The idea.** Let us consider the simplest stratification possible, the stratification that consists of one stratum only. Assume that our space is a simply connected differential manifold. We then know that for real coefficients, $D^b_c(X)$ is described by the dg algebra $\mathrm{End}\, I$, where I is an injective resolution of the constant sheaf $\mathbb{R}_X$. Using that the functor of global sections on X is naturally equivalent to the functor $\mathrm{Hom}(\mathbb{R}_X, \cdot)$ we see that $\mathrm{End}\, I$ is quasi-isomorphic to $R\Gamma(X, \mathbb{R}_X)$. The latter complex calculates the cohomology of the space X, and it is a standard argument of cohomology theory that it is quasi-isomorphic to the complex of global sections of the de Rham complex Ω_X, which in turn in certain circumstances is known to be formal (see, for instance, [DGMS75]).

The de Rham complex is much easier to handle than an arbitrary injective resolution of the constant sheaf. Hence it appears to be a sensible idea to describe $D^b_c(X)$ by differential forms even if the stratification is no longer trivial. We will pursue this idea in this section and deduce a formality result for n-spheres at the end.

3.2. **A generalized de Rham algebra.** Let X be a second-countable differentiable manifold X (which implies that it is para-compact), stratified in a point pt and its complement. Denote by $i_{pt} : X\backslash pt \to X$ the inclusion. Assume that our base ring is the field of real or complex numbers, which we denote by $\mathbb{K}$.

If $X\backslash pt$ is simply connected, we know that $D^b_c(X)$ is generated by the skyscraper $i_{pt!}\mathbb{K}_{pt}$ and the constant sheaf on X, shifted by $\lfloor \frac{\dim X}{2} \rfloor$. However, shifting the constant sheaf does not affect any of the arguments in this section, hence for simplicity of notation we drop the shift, and use the constant sheaf $\mathbb{K}_X$ as generator. We then know from the equivalence (8) that $D^b_c(X)$ is described by the dg algebra $\mathrm{End}(I \oplus i_{pt!}\mathbb{K}_{pt})$, where I is an injective resolution of $\mathbb{K}_X$. Now we would like to find a quasi-equivalent description of $\mathrm{End}(I \oplus i_{pt!}\mathbb{K}_{pt})$ using differential forms. The first problem we need to tackle is that the stratum pt is too small to allow for interesting differential forms. Hence we need to replace it by a small ball to clear some space. Thus for the remainder of this section, fix a closed ball D around pt, i.e. a small closed neighbourhood that is mapped by a chart to a closed ball in $\mathbb{R}^{\dim X}$. Let $i : D \hookrightarrow X$ be the inclusion. By abuse of notation, we use i_{pt} both for the inclusions $pt \hookrightarrow D$ and $pt \hookrightarrow X$.

Proposition 10. *Choose an injective resolution J of the constant sheaf $\mathbb{K}_D$. Then the dg algebras $\mathrm{End}(I \oplus i_{pt!}\mathbb{K}_{pt})$ and $\mathrm{End}(I \oplus i_! J)$ are quasi-equivalent.*

Before we dig into the proof, I would like to recall some standard facts that we are going to use often throughout this section (for proofs see [Ive86]). The first one we have already mentioned: The global section functor $\Gamma(X, \cdot)$ of sheaves of R-modules on a topological space X is naturally equivalent to the functor $\mathrm{Hom}(R_X, \cdot)$. Another useful statement is that, whenever I is a complex of injectives bounded from below, every quasi-isomorphism of complexes $\mathcal{F}$ and $\mathcal{G}$ yields a quasi-isomorphism between $\mathrm{Hom}(\mathcal{F}, I)$ and $\mathrm{Hom}(\mathcal{G}, I)$.

Proof. Consider the operations

$$\mathrm{End}\left(I \oplus i_{pt!}\mathbb{K}_{pt}\right) \circlearrowright \mathrm{Hom}\left(I \oplus i_! J, I \oplus i_{pt!}\mathbb{K}_{pt}\right) \circlearrowleft \mathrm{End}\left(I \oplus i_! J\right)$$

together with the cycle

$$c := \begin{pmatrix} id_I & 0 \\ 0 & \gamma \end{pmatrix} \in \mathrm{Hom}\left(I \oplus i_! J, I \oplus i_{pt!}\mathbb{K}_{pt}\right)$$

where γ is the natural extension of the adjunction morphism $\mathbb{K}_D \to i_{pt!}\mathbb{K}_{pt}$, which is first lifted to J using that the skyscraper is injective, and is then extended by zero.

We need to show that the following maps induce isomorphisms on cohomology:

(1) $\mathrm{Hom}(i_{pt!}\mathbb{K}_{pt}, I) \to \mathrm{Hom}(i_! J, I)$, $a \mapsto a \circ \gamma$
(2) $\mathrm{End}(i_{pt!}\mathbb{K}_{pt}) \to \mathrm{Hom}(i_! J, i_{pt!}\mathbb{K}_{pt})$, $a \mapsto a \circ \gamma$
(3) $\mathrm{Hom}(I, i_! J) \to \mathrm{Hom}(I, i_{pt!}\mathbb{K}_{pt})$, $a \mapsto \gamma \circ a$
(4) $\mathrm{End}(i_! J) \to \mathrm{Hom}(i_! J, i_{pt!}\mathbb{K}_{pt})$, $a \mapsto \gamma \circ a$

Proving this means a lot of diagram chasing, none of which involves unexpected twists. The proofs rely on the fact that the inclusion $X\backslash D \hookrightarrow X\backslash pt$ induces isomorphisms on cohomology, which can be seen using Mayer-Vietoris and the five lemma. We are only going to prove the first statement and leave the remaining three, which are easier, to the reader. Else a complete proof can be found in [Bal06].

For (1), we need to show that the corresponding morphism

$$\mathrm{Ext}^q(i_{pt!}\mathbb{K}_{pt}, \mathbb{K}_X) \to \mathrm{Ext}^q(i_!\mathbb{K}_D, \mathbb{K}_X)$$

is an isomorphism for every $q \in \mathbb{Z}$ (where as usual $\mathrm{Ext}^q(\cdot, \cdot)$ denotes the q^{th} right derived of $\mathrm{Hom}(\cdot, \cdot)$). Consider the open inclusions $j_{pt} : X\backslash pt \hookrightarrow X$ and $j : X\backslash D \hookrightarrow X$ of the complements of pt and D, respectively. We get a commutative diagram of short exact sequences

$$\begin{tikzcd}
j_! j^* \mathbb{K}_X \arrow[r] \arrow[d] & \mathbb{K}_X \arrow[r] \arrow[d, equal] & i_! i^* \mathbb{K}_X \arrow[d] \\
j_{pt!} j_{pt}^* \mathbb{K}_X \arrow[r] & \mathbb{K}_X \arrow[r] & i_{pt!} i_{pt}^* \mathbb{K}_X
\end{tikzcd}$$

where the vertical morphisms are induced by the adjuntions $(j_!, j^*)$ and $(i_pt^*, i_{pt!})$, respectively. But since the inclusion $X\backslash D \hookrightarrow X\backslash pt$ induces isomorphisms on cohomology, we get that concatenation with the left vertical yields isomorphisms

$$\mathrm{Ext}^q(j_{pt!} j_{pt}^* \mathbb{K}_X, \mathbb{K}_X) \xrightarrow{\sim} \mathrm{Ext}^q(j_! j^* \mathbb{K}_X, \mathbb{K}_X)$$

Applying the functor $\mathrm{Ext}(\cdot, \mathbb{K}_X)$ to our pair of short exact sequences yields the corresponding commutative ladder, from which the required result follows by applying the five lemma. $\square$

Next, we would like to describe the dg algebra $\operatorname{End}(I \oplus i_! J)$ using differential forms. Denote by Ω the sheaf of $\mathbb{K}$-valued differential forms on X, which is a soft resolution of the constant sheaf. In addition to the inclusion $i : D \hookrightarrow X$ we now need the inclusion $o : int(D) \hookrightarrow X$, where $int(D)$ denotes the open interior of D. To avoid notational inflation we write $\Gamma\mathcal{F}$ for the global sections of a sheaf $\mathcal{F}$ on X. We would now like to consider the differential graded matrix algebra

$$\mathcal{M}(\Omega) := \begin{pmatrix} \Gamma\Omega & \Gamma o_! o^*\Omega \\ \Gamma i_! i^*\Omega & \Gamma i_! i^*\Omega \end{pmatrix} \tag{11}$$

whose multiplicative structure is induced by the wedge product. How this generalized wedge product on $\mathcal{M}(\Omega)$ works should be clear if we think of $\Gamma o_! o^*\Omega$ as the sections of Ω with support in $int(D)$ and of $\Gamma i_! i^*\Omega$ as the sections over D, which can be extended to X since the sheaf of differential forms Ω is soft.

Using the same point of view we see that there is a natural inclusion

$$\begin{pmatrix} \Gamma\Omega & \Gamma o_! o^*\Omega \\ \Gamma i_! i^*\Omega & \Gamma i_! i^*\Omega \end{pmatrix} \hookrightarrow \begin{pmatrix} \operatorname{End}\Omega & \operatorname{Hom}(i_! i^*\Omega, \Omega) \\ \operatorname{Hom}(\Omega, i_! i^*\Omega) & \operatorname{End} i_! i^*\Omega \end{pmatrix}.$$

This inclusion is again induced by the wedge product, more precisely by the map $a \mapsto a \wedge \cdot$. Since this inclusion is obviously compatible with the multiplicative structures, we have proven that $\mathcal{M}(\Omega)$ has a canonical structure of dg-sub-algebra of

$$\begin{pmatrix} \operatorname{End}\Omega & \operatorname{Hom}(i_! i^*\Omega, \Omega) \\ \operatorname{Hom}(\Omega, i_! i^*\Omega) & \operatorname{End} i_! i^*\Omega \end{pmatrix} = \operatorname{End}(\Omega \oplus i_! i^*\Omega).$$

Proposition 12. *For any injective resolution J of the constant sheaf $\mathbb{K}_D$, the dg algebra $\mathcal{M}(\Omega)$ operates as a dg-sub-algebra of $\operatorname{End}(\Omega \oplus i_! i^*\Omega)$ canonically on $\operatorname{Hom}(\Omega \oplus i_! i^*\Omega, I \oplus i_! J)$. This operations yields a quasi-equivalence between $\mathcal{M}(\Omega)$ and $\operatorname{End}(I \oplus i_! J)$.*

Proof. The morphisms $\mathbb{K}_X \hookrightarrow I$ and $\mathbb{K}_D \hookrightarrow J$ can be lifted in the homotopy category to maps $\bar{\imath} : \Omega \hookrightarrow I$ and $\bar{\jmath} : i^*\Omega \hookrightarrow J$. We get a quasi-isomorphism

$$c := \begin{pmatrix} \bar{\imath} & 0 \\ 0 & i_! \bar{\jmath} \end{pmatrix} \in \operatorname{Hom}(\Omega \oplus i_! i^*\Omega, I \oplus i_! J)$$

which is going to yield our quasi-equivalence. A standard argument shows that

$$\operatorname{End}(I \oplus i_! J) \to \operatorname{Hom}\left(\Omega \oplus i_! i^*\Omega, I \oplus i_! J\right),\ a \mapsto a \circ c$$

is a quasi-isomorphism, and it remains to show that the same is true for

$$\begin{pmatrix} \Gamma\Omega & \Gamma o_! o^*\Omega \\ \Gamma i_! i^*\Omega & \Gamma i_! i^*\Omega \end{pmatrix} \to \operatorname{Hom}\left(\Omega \oplus i_! i^*\Omega, I \oplus i_! J\right),\ b \mapsto c \circ b$$

For this, we need to show that the following maps are quasi-isomophisms:

(1) $\Gamma\Omega \to \operatorname{Hom}(\Omega, I)$, $a \mapsto \bar{\imath} \circ (a \wedge \cdot)$
(2) $\Gamma o_! o^*\Omega \to \operatorname{Hom}(i_! i^*\Omega, I)$, $a \mapsto \bar{\imath} \circ (a \wedge \cdot)$
(3) $\Gamma i_! i^*\Omega \to \operatorname{Hom}(\Omega, i_! J)$, $a \mapsto i_! \bar{\jmath} \circ (a \wedge \cdot)$
(4) $\Gamma i_! i^*\Omega \to \operatorname{Hom}(i_! i^*\Omega, i_! J)$, $a \mapsto i_! \bar{\jmath} \circ (a \wedge \cdot)$

(1) A quick one as starter: Consider the commutative diagram

$$\begin{tikzcd} \Gamma\Omega \arrow[r] \arrow[d] & \Gamma I \arrow[d, "\wr"] \\ \operatorname{Hom}(\Omega, I) \arrow[r] & \operatorname{Hom}(\mathbb{K}_X, I) \end{tikzcd}$$

It is a standard result that the horizontal maps are quasi-isomorphism (for the upper one we need that Ω is a soft, hence Γ-acyclic resolution of the constant sheaf), as well as that the right vertical is an isomorphism. Hence the left map must have been a quasi-isomorphism, too, and we are done.

(2) We would like to remind the reader of the fact that for any sheaf $\mathcal{F}$ on X, $\Gamma o_! o^* \mathcal{F}$ are the sections with support in $int(D)$, and $\Gamma i_! i^{(!)} \mathcal{F}$ are the sections with support in D (where $i^{(!)}$ is the right adjoint of $i_!$ as explained in section 2.1). We get a natural inclusion

$$\Gamma o_! o^* \mathcal{F} \hookrightarrow \Gamma i_! i^{(!)} \mathcal{F}$$

which yields the upper right horizontal map in the following diagram,

$$\begin{tikzcd}
\Gamma o_! o^* \Omega \arrow[r] \arrow[d] & \Gamma o_! o^* I \arrow[r] & \Gamma i_! i^{(!)} I \arrow[d, "\wr"] \\
\mathrm{Hom}(i_! i^* \Omega, \Omega) \arrow[d] & & \mathrm{Hom}(\mathbb{K}_X, i_! i^{(!)} I) \arrow[d, "\wr"] \\
\mathrm{Hom}(i_! i^* \Omega, I) \arrow[rr] & & \mathrm{Hom}(i_! i^* \mathbb{K}_X, I)
\end{tikzcd}$$

where all other maps should be obvious. As always, the two right verticals are isomorphisms (the lower one uses two adjunctions), and the lower horizontal is a quasi-isomorphism. The functors $o_!$ and o^* are exact, hence $o_! o^* \Omega \to o_! o^* I$ is a quasi-isomorphism, and since soft sheaves on paracompact spaces are Γ-acyclic, it induces a quasi-isomorphism $\Gamma o_! o^* \Omega \to \Gamma o_! o^* I$. Thus it remains to prove that the natural map $\Gamma o_! o^* I \to \Gamma i_! i^{(!)} I$ is a quasi-isomorphism.

For this, we need to calculate the cohomologies of both sides. It is easy to check that for a locally closed subset $Z \overset{z}{\hookrightarrow} Y$ of a topological space Y, such that Z has compact closure, we have a natural isomorphism of functors $\Gamma(Y, \cdot) \circ z_! \simeq \Gamma_c(Z, \cdot)$. Since $o^* I$ is an injective resolution of the constant sheaf $\mathbb{K}_{intD}$ on $int(D)$, we get

$$H^q(\Gamma o_! o^* I) = H^q_c(int(D), \mathbb{K}_{int(D)}) \simeq \begin{cases} 0, \ q \neq \dim_{\mathbb{R}} X \\ \mathbb{K}, \ q = \dim_{\mathbb{R}} X \end{cases}$$

where the last isomorphism is due to Poincaré duality.

The q^{th} cohomology of $\Gamma i_! i^{(!)} I$ is, as follows from the above diagram, equal to $\mathrm{Ext}^q(i_! i^* \mathbb{K}_X, \mathbb{K}_X)$, which in turn is, as we have seen in the proof of Proposition 10, $\mathrm{Ext}^q(i_{pt!} \mathbb{K}_{pt}, \mathbb{K}_X)$. A standard argument shows that $i^!_{pt} \mathbb{K}_X \simeq \mathbb{K}_{pt}[-\dim X]$, and we get

$$\mathrm{Ext}^q(i_{pt!} \mathbb{K}_{pt}, \mathbb{K}_X) = \mathrm{Ext}^q(\mathbb{K}_{pt}, i^!_{pt} \mathbb{K}_X) \simeq \begin{cases} 0, \ q \neq \dim_{\mathbb{R}} X \\ \mathbb{K}, \ q = \dim_{\mathbb{R}} X \end{cases}$$

So we only need to check that the morphism $\Gamma o_! o^* I \to \Gamma i_! i^* I$ induces a surjection on the $(\dim X)^{\text{th}}$ cohomology. For this, choose a smaller ball $\hat{D}$ that is contained in the interior of D, and denote by $\hat{\imath} : \hat{D} \hookrightarrow X$ the inclusion. We see that the inclusion $\Gamma \hat{\imath}_! \hat{\imath}^* I \to \Gamma i_! i^* I$ is a quasi-isomorphism. (This can be seen as in the proof of Proposition 10, where we saw that both are quasi-isomorphic to $\mathrm{Hom}(i_{pt!} i^*_{pt} \mathbb{K}_X, I)$ via compatible quasi-isomorphisms.) However, the quasi-isomorphism $\Gamma \hat{\imath}_! \hat{\imath}^* I \to \Gamma i_! i^* I$ splits over $\Gamma o_! o^* I$, hence the inclusion $\Gamma o_! o^* I \to \Gamma i_! i^* I$ must induce surjections on cohomology.

(3) Considering the following diagram

$$\begin{tikzcd} \Gamma i_! i^*\Omega \arrow[r] \arrow[d] & \Gamma i_! J \arrow[d, "\wr"] \\ \mathrm{Hom}(\Omega, i_! J) \arrow[r] & \mathrm{Hom}(\mathbb{K}_X, i_! J) \end{tikzcd}$$

we can use a similar argument to (1).

(4) For this we use the diagram

$$\begin{tikzcd} \Gamma i_! i^*\Omega \arrow[r] \arrow[dd] & \Gamma i_! J \arrow[d, "\wr"] \\ & \mathrm{Hom}(\mathbb{K}_X, i_! J) \arrow[d, "\wr"] \\ \mathrm{Hom}(i_! i^*\Omega, i_! J) \arrow[r] & \mathrm{Hom}(i_! i^*\mathbb{K}_X, i_! J) \end{tikzcd}$$

and again a similar argument. □

Combining the equivalence (8), Proposition 10 and Proposition 12, we can summarize the results from this section in the following theorem:

Theorem 13. *Let X be a second countable differentiable manifold and pt a point in X such that $X \backslash pt$ is simply connected. For Ω the sheaf of $\mathbb{K}$-valued differential forms and D any closed ball around pt, let $\mathcal{M}(\Omega)$ be as defined in (11). Then we have an equivalence of triangulated categories*

$$D^b_c(X) \simeq \mathcal{D}^f_{\mathcal{M}(\Omega)}$$

where $\mathcal{D}^f_{\mathcal{M}(\Omega)}$ is the subcategory of $(Per_{\mathcal{M}(\Omega)})^\circ$ corresponding to $\mathcal{D}^f_{\mathrm{End}\,(I \oplus i_{pt!}\mathbb{K}_{pt})}$ under the equivalence of categories

$$Per_{\mathrm{End}\,(I \oplus i_{pt!}\mathbb{K}_{pt})} \simeq Per_{\mathcal{M}(\Omega)}$$

which is induced by the quasi-equivalence of the dg algebras $\mathrm{End}(I \oplus i_{pt!}\mathbb{K}_{pt})$ and $\mathcal{M}(\Omega)$.

A short remark to finish: For the sake of readability we have refrained from allowing rings as coefficients in this section. However, the reader can check that all statements in this section hold for rings as well, if we replace the corresponding skyscraper $i_{pt!}R_{pt}$ by an injective resolution, and the sheaf of differential forms by the sheaf of singular cochains (or indeed by any other soft resolution of the constant sheaf by a sheaf of dg algebras whose restrictions to open subsets are again soft). This allows us to take coefficients in a PID, and with little more effort, get an analogous result.

3.3. **Formality for spheres.** Denote by S^n the sphere in $\mathbb{R}^{n+1}$. For $n \geq 2$, we consider the stratification of S^n consisting of a point pt and its complement, and now use Theorem 13 to prove formality of the corresponding constructible derived category.

We use the notations from the previous section: For $\mathbb{K}$ the real or complex numbers, let Ω be the sheaf of $\mathbb{K}$-valued differential forms on S^n. Choose a small

closed ball D around pt. For $i : D \hookrightarrow S^n$ and $o : int(D) \hookrightarrow S^n$ the inclusions, let $\mathcal{M}(\Omega)$ be the dg algebra defined in (11). Theorem 13 then provides an equivalence

$$D^b_c(S^n) \simeq \mathcal{D}^f_{\mathcal{M}(\Omega)}$$

and we now give the promised formality result:

Theorem 14. *The generalized de Rham algebra $\mathcal{M}(\Omega)$ on the n-sphere S^n (for $n \geq 2$) is formal.*

Proof. First we need to understand what the cohomology of $\mathcal{M}(\Omega)$ looks like. The global sections $\Gamma\Omega$ calculate the cohomology of the sphere:

$$H^q(\Gamma\Omega) = \begin{cases} \mathbb{K}, & q = n \text{ or } 0 \\ 0, & \text{otherwise} \end{cases}$$

As in the proof of Proposition 12, we see that

$$H^q(\Gamma o_! o^* \Omega) = H^q_c(int(D), \mathbb{K}_{int(D)}) \simeq \begin{cases} \mathbb{K}, & q = n \\ 0, & q \neq n \end{cases}$$

and

$$H^q(\Gamma i_! i^* \Omega) = H^q(D, \mathbb{K}_D) \simeq \begin{cases} \mathbb{K}, & q = 0 \\ 0, & q \neq 0 \end{cases} \quad .$$

We claim that the natural inclusion $\Gamma o_! o^* \Omega \to \Gamma\Omega$ yields an isomorphism on the n^{th} cohomology group. This can be seen as follows: Denoting by $h : S^n \backslash int(D) \hookrightarrow S^n$ the inclusion, we get a short exact sequence

$$o_! o^* \Omega \hookrightarrow \Omega \twoheadrightarrow h_! h^* \Omega$$

Since all of those are complexes of soft, hence Γ-acyclic sheaves, we get a corresponding short exact sequence on the global sections. Our claim follows from the corresponding long exact cohomology sequence, if we use that the cohomology groups

$$H^q(\Gamma h_! h^* \Omega) = H^q(\Gamma(S^n \backslash int(D), h^*\Omega)) = H^q(S^n \backslash int(D), \mathbb{K}_{S^n \backslash int(D)})$$

vanish for $q > 0$.

Now choose a generator $\omega \in \Gamma o_! o^* \Omega$ of the n^{th} cohomology. By what we have just shown, ω must then be a generator of the n^{th} cohomology of the sphere S^n. Since $\Gamma i_! i^* \Omega$ has no n^{th} cohomology, the cohomology class of $\omega|_D$ must vanish. Hence there is a $\tau \in \Gamma i_! i^* \Omega^{n-1}$ satisfying $d\tau = \omega|_D$.

Now define a sub-vector space $\mathcal{U}$ of $\mathcal{M}(\Omega)$ as follows:

$$\mathcal{U} := \begin{pmatrix} \begin{matrix} \mathbb{K}\cdot\omega \\ \oplus \\ \mathbb{K}\cdot 1 \end{matrix} & \mathbb{K}\cdot\omega \\ & \\ \begin{matrix} \mathbb{K}\cdot\omega|_D \\ \oplus \\ \mathbb{K}\cdot\tau \\ \oplus \\ \mathbb{K}\cdot 1 \end{matrix} & \begin{matrix} \mathbb{K}\cdot\omega|_D \\ \oplus \\ \mathbb{K}\cdot\tau \\ \oplus \\ \mathbb{K}\cdot 1 \end{matrix} \end{pmatrix}$$

where $\mathbb{K} \cdot \omega$ denotes the vector space generated by ω etc. It is straightforward that $\mathcal{U}$ is closed under multiplication: As the de Rham-complex vanishes in degrees greater than n, it suffices to check that $\tau \wedge \tau = 0$. For $n \geq 3$ this is clear for degree reasons, and for $n = 2$ it holds because $\Gamma i_! i^* \Omega$ is super-commutative and the degree of τ is odd.

From the construction of $\mathcal{U}$ it is clear that the inclusion $\mathcal{U} \hookrightarrow \mathcal{M}(\Omega)$ is a quasi-isomorphism. It is also obvious that the projection $\mathcal{U} \twoheadrightarrow H(\mathcal{U})$ is a quasi-isomorphism, hence the chain of quasi-isomorphisms

$$\mathcal{M}(\Omega) \hookleftarrow \mathcal{U} \twoheadrightarrow H(\mathcal{U})$$

yields the desired formality. □

As mentioned at the beginning of this section, it is straightforward to check that shifting the constant sheaf on S^n does not affect any of the arguments in this section. All we need to do is adapt the definition of $\mathcal{M}(\Omega)$ by introducing corresponding shifts. Hence in this section we have effectively proved Theorem 3 (ii) from the introduction.

4. A description of $D^b_c(X)$ by representations of quivers

4.1. $\mathcal{S}$-acyclic sheaves and $\mathcal{S}$-acyclic stratifications. In this section we would like to describe $D^b_c(X)$ using representations of quivers. Let R be a commutative unitary Noetherian ring of finite homological dimension and X a stratified topological space. We would like to remind the reader that we assume any stratification to satisfy the conditions (i) - (iii) introduced in section 2.2. We denote by $\mathcal{S}$ the set of strata, and for a stratum $S \in \mathcal{S}$ by Et_S its star, i.e. the union of all strata whose closure contains S:

$$\mathrm{Et}_S = \bigcup_{S \subset \overline{T}} T$$

The star of S is the smallest open set consisting of strata that contains S.

Let us now consider the projection $p : X \twoheadrightarrow \mathcal{S}$ that maps every point to the stratum that contains it. We endow $\mathcal{S}$ with the final topology and would now like to link the category of sheaves on $\mathcal{S}$, $\mathrm{Sh}(\mathcal{S})$, to the category of weakly constructible sheaves on X, $\mathrm{Sh}_{w,\,c}(X)$.

Proposition 15. *Let $\mathcal{S}$ be a stratification of a topological space X that satisfies the following assumption: For every weakly constructible sheaf $\mathcal{F}$, every stratum $S \in \mathcal{S}$ and every $x \in S$ the natural morphism*

$$\Gamma(Et_S, \mathcal{F}) \to \mathcal{F}_x$$

is an isomorphism. Then for $p : X \twoheadrightarrow \mathcal{S}$ the projection, the adjoint pair (p^*, p_*) yields an equivalence of categories

$$\mathrm{Sh}_{w,\,c}(X) \underset{p^*}{\overset{p_*}{\rightleftarrows}} \mathrm{Sh}(\mathcal{S})$$

Proof. We just need to check that $p^* p_* \to id$ and $id \to p_* p^*$ are natural isomorphisms, which is straightforward by looking at the stalks. □

We would like to extend this statement to the corresponding derived categories. For this, we need to introduce the notions of $\mathcal{S}$-acyclic sheaves and acyclic stratifications.

Definition 16. Let $\mathcal{S}$ be a stratification of a topological space X. We call a sheaf $\mathcal{F}$ on X $\mathcal{S}$-acyclic if, for every stratum $S \in \mathcal{S}$ and every $i > 0$,

$$H^i\big(\mathrm{Et}_S, \mathcal{F}\big) = 0$$

holds. This is equivalent to $\mathcal{F}$ being p_*-acyclic, where $p : X \to \mathcal{S}$ is the projection.

Definition 17. We call a stratification $\mathcal{S}$ of X acyclic if it satisfies the following two conditions:

(1) For every weakly constructible sheaf $\mathcal{F}$, every stratum $S \in \mathcal{S}$ and and every $x \in S$ the natural map $\Gamma(\mathrm{Et}_S, \mathcal{F}) \to \mathcal{F}_x$ is an isomorphism.
(2) Every weakly constructible sheaf on X is $\mathcal{S}$-acyclic.

It is proved in [KS94] that simplicial complexes with their natural stratification are acyclic. We will see more examples of acyclic stratifications in section 5.

Theorem 18. *On a topological space X with an acyclic stratification $\mathcal{S}$ the adjoint pair (Rp_*, p^*) induces an equivalence of categories*

$$D^+_{w,\,c}(X) \underset{p^*}{\overset{Rp_*}{\rightleftarrows}} D^+(\mathcal{S}).$$

If X has finite homological dimension, these induce equivalences

$$D^b_{w,\,c}(X) \underset{p^*}{\overset{Rp_*}{\rightleftarrows}} D^b(\mathcal{S}).$$

Proof. We have to show that $id \to Rp_*p^*$ and $p^*Rp_* \to id$ are natural equivalences. The first case follows from Proposition 15, and the fact that p^* transforms a sheaf on $\mathcal{S}$ into a p_*-acyclic object. For the second case we need to show that for any complex $\mathcal{F} \in D^+_{w,\,c}(X)$ consisting of p^*p_*-acyclic objects the morphism $p^*p_*\mathcal{F} \to \mathcal{F}$ is a quasi-isomorphism. This, however, works exactly as the proof of Proposition 8.1.9 in [KS94]. $\square$

Unfortunately, the same statement for the constructible derived category doesn't work quite as smoothly. We need to introduce some notation first: Let $\mathcal{C}$ be an abelian category and $\mathcal{C}'$ a thick full abelian subcategory. Then $D^+_{\mathcal{C}'}(\mathcal{C})$ denotes the full triangulated subcategory of $D^+(\mathcal{C})$ consisting of complexes whose cohomology objects are in $\mathcal{C}'$. $D^b_{\mathcal{C}'}(\mathcal{C})$ is defined analogously. We are interested in $\mathcal{C} = \mathrm{Sh}(\mathcal{S})$, the category of sheaves on $\mathcal{S}$, and $\mathcal{C}' = \mathrm{Sh}_{\mathrm{f}}(\mathcal{S})$, the full abelian subcategory of sheaves with finitely generated stalks.

Theorem 19. *On a topological space X with an acyclic stratification $\mathcal{S}$ the adjoint pair (Rp_*, p^*) induces an equivalence of categories*

$$D^+_c(X) \underset{p^*}{\overset{Rp_*}{\rightleftarrows}} D^+_{\mathrm{Sh}_{\mathrm{f}}(\mathcal{S})}(\mathrm{Sh}(\mathcal{S})).$$

If X has finite homological dimension, these induce equivalences

$$D^b_c(X) \underset{p^*}{\overset{Rp_*}{\rightleftarrows}} D^b_{\mathrm{Sh}_{\mathrm{f}}(\mathcal{S})}(\mathrm{Sh}(\mathcal{S})).$$

Proof. The proof works as for Theorem 18. Note that this theorem is still true if we allow stratifications that are only "almost" acyclic: The second condition in the definition of an acyclic stratification needs to be true only for constructible sheaves. $\square$

This theorem is rounded off by the following statement:

Proposition 20. *For any stratified topological space the natural functor*

$$D^b(\mathrm{Sh_f}(\mathcal{S})) \quad \xrightarrow{\sim} \quad D^b_{\mathrm{Sh_f}(\mathcal{S})}(\mathrm{Sh}(\mathcal{S}))$$

yields an equivalence of triangulated categories.

Proof. The trick is to use a standard criterion for an equivalence between $D^b(\mathcal{C})$ and $D^b_{\mathcal{C}'}(\mathcal{C})$, which can be found in [KS94]. The details work very similarly to the proof of Theorem 8.1.11 in [KS94]; a detailed proof can be found in [Bal06]. □

4.2. **Representations of quivers.** We would now like to make the transition from sheaves on $\mathcal{S}$ to representations of quivers. A detailed account on quivers can be found in [GR97]; I will only give a very short summary here.

By a quiver we mean a finite directed graph. A representation V of a quiver is a collection of R-modules V_x for every vertex x (the stalks), together with R-linear maps $V_\gamma : V_x \to V_y$ for every arrow γ from x to y. A map of representations is a compatible collection of maps on the stalks. For a quiver Q we denote by $\mathrm{Rep}\, Q$ the category of representations of Q and by $\mathrm{Rep_f}\, Q$ the category of representations whose stalks are finitely generated. A relation in a quiver is an R-linear combination of paths starting and ending at the same vertices. If we have a quiver Q with a set of relations ρ, we denote by $\mathrm{Rep}(Q, \rho)$ the full subcategory of $\mathrm{Rep}\, Q$ given by representations that are compatible with the relations. The category $\mathrm{Rep_f}(Q, \rho)$ is defined analogously.

To every stratified space $(X, \mathcal{S})$ we assign a quiver as follows: The vertices are given by the strata in $\mathcal{S}$, and there is an arrow from S to T if and only if $S \subset \overline{T}$. By identifying all paths between two vertices we get a set of relations. We denote the corresponding quiver with relations by $(Q_\mathcal{S}, \rho_\mathcal{S})$.

Now there is a quite obvious functor $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Rep}(Q_\mathcal{S}, \rho_\mathcal{S})$ and vice versa: From a sheaf $\mathcal{F}$ on $\mathcal{S}$ we get a representation by assigning to each vertex the corresponding stalk, and to an arrow $S \to T$ we assign the restriction map $\Gamma(p(\mathrm{Et}_S), \mathcal{F}) \to \Gamma(p(\mathrm{Et}_T), \mathcal{F})$. This functor is exact and induces equivalences $\mathrm{Rep}(Q_\mathcal{S}, \rho_\mathcal{S}) \xrightarrow{\sim} \mathrm{Sh}(\mathcal{S})$ and $\mathrm{Rep_f}(Q_\mathcal{S}, \rho_\mathcal{S}) \xrightarrow{\sim} \mathrm{Sh_f}(\mathcal{S})$. This is easily checked using the obvious inverse functor, which is also exact. Hence for $* = +$ or b, these functors yield equivalences

$$D^*(\mathrm{Sh}(\mathcal{S})) \rightleftarrows D^*(\mathrm{Rep}(Q_\mathcal{S}, \rho_\mathcal{S})) \qquad \text{and} \qquad D^*(\mathrm{Sh_f}(\mathcal{S})) \rightleftarrows D^*(\mathrm{Rep_f}(Q_\mathcal{S}, \rho_\mathcal{S})).$$

Summarizing the results from this section, we get the following theorem:

Theorem 21. *We can assign a quiver $Q_\mathcal{S}$ with relations $\rho_\mathcal{S}$ to every acyclic stratification $\mathcal{S}$ on a topological space X of finite homological dimension, such that we get natural equivalences*

$$D^*_{w,c}(X) \xrightarrow{\sim} D^*(\mathrm{Rep}(Q_\mathcal{S}, \rho_\mathcal{S})) \qquad \textit{for } * = +, b$$

and

$$D^b_c(X) \xrightarrow{\sim} D^b(\mathrm{Rep_f}(Q_\mathcal{S}, \rho_\mathcal{S}))$$

5. Formality for the 2-sphere

We would now like to prove formality for the 2-sphere S^2 stratified in several points and their complement. The clue is to use Theorem 21 to transfer the problem of formality to a category of representations of quivers. Since we are going to encounter several stratifications during this chapter, we are going to carry the

relevant stratification in the notation of the constructible derived category; hence from now on we will write $D^b_{c,\mathcal{S}}(X)$ instead of $D^b_c(X)$. For the remainder of this paper we assume the base ring R to be a principal ideal domain.

Denote by $\mathfrak{S}_n$ the stratification of the 2-sphere in n points and their complement (this stratification is not unique, but it does not matter for our purposes which points we choose). $\mathfrak{S}_0$ denotes the trivial stratification. We will need to "acyclify" those stratifications; hence we would now like to introduce certain acyclic stratifications of the 2-sphere: Take n points ($n \geq 2$) on a great circle on the sphere, and stratify the sphere into those points, the circle segments between them and the two hemi-spheres. We denote this stratification by $\mathfrak{A}_n$. This stratification is acyclic; the proof is not difficult and can be found in [Bal06]. Obviously it is not crucial that the points are on a great circle; if they are not, it is still be possible to connect them via a deformed circle, and the resulting stratification will still be acyclic.

5.1. **The trivial stratification.** First, we would like to consider the trivial stratification $\mathfrak{S}_0$ on S^2 and the corresponding constructible derived category $D^b_{c,\mathfrak{S}_0}(S^2)$. We know that this category is generated by the constant sheaf R_{S^2}. It follows from (8) that for any injective resolution I of the constant sheaf, $D^b_{c,\mathfrak{S}_0}(S^2)$ is equivalent to $\mathcal{D}^f_{\operatorname{End} I}$. We will now show that the dg algebra $\operatorname{End} I$ is formal:

Proposition 22. *Let I be an injective resolution of the constant sheaf R_{S^2} on the 2-sphere. Then $\operatorname{End} I$ is quasi-equivalent to $R[t]/t^2$, where t lives in degree 2. In particular, $\operatorname{End} I$ is formal.*

The standard proof for this statement would exploit the fact that $\operatorname{End} I$ calculates the cohomology of the 2-sphere, and to directly give a quasi-isomorphism $H(\operatorname{End} I) \to \operatorname{End} I$. However, we would like to give a different proof as a preparation for more complex cases. We will break down the proof into several statements; the crucial part is Proposition 26.

First we need to acyclify the trivial stratification. For this we take the stratification $\mathfrak{A}_2$ (as introduced at the beginning of section 5), which consists of two points, two hemi-equators and two hemi-spheres. We denote the two points by P_1 and P_2, the hemi-equators by E_1 and E_2, and the hemi-spheres by H_1 and H_2.

The quiver $(Q_{\mathfrak{A}_2}, \rho_{\mathfrak{A}_2})$ associated to this stratification, as introduced in section 4.2, is given by

$$\begin{tikzcd}
\bullet & \bullet \\
\bullet \arrow[u] \arrow[ur] & \bullet \arrow[u] \arrow[ul] \\
\bullet \arrow[u] \arrow[ur] & \bullet \arrow[u] \arrow[ul]
\end{tikzcd}$$

with relations that identify all paths that have the same starting and end point. In the diagram above, the two upper vertices correspond to the hemi-spheres, the middle ones to the hemi-equators and the lower ones to the two points in the stratification. By definition of $(Q_{\mathfrak{A}_2}, \rho_{\mathfrak{A}_2})$ we should also have arrows from the lower to the upper vertices; however, due to the relations we can drop those.

We would now like to transfer our problem to a category of representations of this quiver, using the natural equivalences of categories

$$\operatorname{Rep}(Q_{\mathfrak{A}_2}, \rho_{\mathfrak{A}_2}) \xrightarrow{\sim} \operatorname{Sh}(\mathfrak{A}_2) \xrightarrow{\sim} \operatorname{Sh}_{w,\,c,\,\mathfrak{A}_2}(S^2)$$

that we constructed in the previous section. Under these equivalences, the representation corresponding to the constant sheaf R_{S^2} is

$$C \quad := \quad \begin{tikzcd} R & R \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \end{tikzcd}$$

with all arrows carrying the identity map.

Lemma 23. *Let J be a bounded resolution of the representation C that is sufficiently acyclic in the sense that $\operatorname{Ext}^q(J^m, J^n) = 0$ for all $q > 0$ and all $m, n \in \mathbb{Z}$. Then our dg algebra $\operatorname{End} I$ from Proposition 22 is quasi-equivalent to the dg algebra $\operatorname{End} J$.*

Proof. Using that the concatenation $\operatorname{Rep}(Q_{\mathfrak{A}_2}, \rho_{\mathfrak{A}_2}) \to \operatorname{Sh}(\mathfrak{A}_2) \to \operatorname{Sh}_{w,\,c,\,\mathfrak{A}_2}(S^2)$ is exact and fully faithful, the result follows by applying Proposition 9. □

The next thing to do is resolving C by sufficiently acyclic objects. As sufficiently acyclic objects, we would like to take, for every stratum S of $\mathfrak{A}_2$, the representation I_S corresponding to the constant sheaf along the closure of S. To be more precise, for $i : \overline{S} \hookrightarrow S^2$ the inclusion, I_S is the image of $i_! R_{\overline{S}}$ under the equivalence $\operatorname{Sh}_{w,c,\mathfrak{A}_2} \xrightarrow{\sim} \operatorname{Rep}(Q_{\mathfrak{A}_2}, \rho_{\mathfrak{A}_2})$. This procedure yields the following representations of $(Q_{\mathfrak{A}_2}, \rho_{\mathfrak{A}_2})$:

$$I_{H_1} = \begin{tikzcd} R & 0 \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \end{tikzcd} \qquad I_{H_2} = \begin{tikzcd} 0 & R \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \end{tikzcd}$$

$$I_{E_1} = \begin{tikzcd} 0 & 0 \\ R \arrow[u] \arrow[ur] & 0 \arrow[u] \arrow[ul] \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \end{tikzcd} \qquad I_{E_2} = \begin{tikzcd} 0 & 0 \\ 0 \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \end{tikzcd}$$

$$I_{P_1} = \begin{tikzcd} 0 & 0 \\ 0 \arrow[u] \arrow[ur] & 0 \arrow[u] \arrow[ul] \\ R \arrow[u] \arrow[ur] & 0 \arrow[u] \arrow[ul] \end{tikzcd} \qquad I_{P_2} = \begin{tikzcd} 0 & 0 \\ 0 \arrow[u] \arrow[ur] & 0 \arrow[u] \arrow[ul] \\ 0 \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \end{tikzcd}$$

where on the arrows we have the identity on R, wherever possible, and the the zero map everywhere else. These objects are indeed sufficiently acyclic:

Lemma 24. *For arbitrary strata $S, T \in \mathfrak{A}_2$ and every $q > 0$ we have that*

$$\operatorname{Ext}^q(I_S, I_T) = 0.$$

Proof. It suffices to show the same statement for the corresponding sheaves. The explicit arguments can be found in [Bal06]. $\square$

Next, we need to consider the morphisms between our sufficiently acyclic objects. The morphism spaces between the various I_S are free R-modules. They vanish in all cases but the following, in which they are free of rank 1:

$$\begin{array}{ll}
\operatorname{Hom}(I_{H_j}, I_{H_j}) & \text{for } j = 1, 2; \\
\operatorname{Hom}(I_{H_j}, I_{E_i}) & \text{for } i = 1, 2 \text{ und } j = 1, 2; \\
\operatorname{Hom}(I_{H_j}, I_{P_i}) & \text{for } i = 1, 2 \text{ und } j = 1, 2; \\
\operatorname{Hom}(I_{E_i}, I_{E_i}) & \text{for } i = 1, 2; \\
\operatorname{Hom}(I_{E_j}, I_{P_i}) & \text{for } i = 1, 2 \text{ und } j = 1, 2; \\
\operatorname{Hom}(I_{P_i}, I_{P_i}) & \text{for } i = 1, 2.
\end{array}$$

Each of these morphism spaces has a canonical generator, which is given by the identity on R, wherever possible, and zeros everywhere else. The generator of $\operatorname{Hom}(I_{H_i}, I_{H_i})$ is denoted by h_i, analogously we denote by e_i and p_i the generators of $\operatorname{Hom}(I_{E_i}, I_{E_i})$ and $\operatorname{Hom}(I_{P_i}, I_{P_i})$, respectively. The generator of $\operatorname{Hom}(I_{H_j}, I_{E_i})$ is denoted by h^e_{ji}, that of $\operatorname{Hom}(I_{H_j}, I_{P_i})$ by h^p_{ji} and that of $\operatorname{Hom}(I_{E_j}, I_{P_i})$ by e_{ji}. [1]

Consider now the following resolution J of C:

$$C \hookrightarrow I_{H_1} \oplus I_{H_2} \xrightarrow{\begin{pmatrix} h^e_{11} & -h^e_{21} \\ h^e_{12} & -h^e_{22} \end{pmatrix}} I_{E_1} \oplus I_{E_2} \xrightarrow{\begin{pmatrix} e_{11} & -e_{21} \\ -e_{12} & e_{22} \end{pmatrix}} I_{P_1} \oplus I_{P_2} \tag{25}$$

Exactness of the sequence is easily checked on stalks.

Proposition 26. *The endomorphism-dg-algebra* $\operatorname{End} J$ *of the above resolution of C is quasi-isomorphic to its cohomology.*

This result is the essential step in the proof of Proposition 22: By Lemma 23, $\operatorname{End} I$ is quasi-equivalent to $\operatorname{End} J$, which in turn is quasi-isomorphic to its cohomology by the above proposition, and the cohomology is easily seen to be $R[t]/t^2$ (where t lives in degree 2). So it remains to prove Proposition 26.

Proof. This proof is based on a hands-on calculation of $\operatorname{End} J$ and its cohomology. For shortness of notation we write $\mathcal{E}$ for $\operatorname{End} J$. Since $\operatorname{Hom}(\cdot, \cdot)$ commutes in both entries with finite direct sums, we get that for any $m \in \mathbb{Z}$, $\mathcal{E}^m$ is a direct sum of free R-modules, which we would now like to give a basis of. Before we get down to this, however, we would like to fix the following notational conventions: Morphisms between direct sums are written as matrices, as is the usual convention. However, matrices of the form $\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & g \\ 0 & 0 \end{pmatrix}$ etc. are abbreviated to g, as are tupels of the

[1] The underlying principle of these notations is as follows: If a morphism starts at I_{H_j}, we use h for notation, if it starts at I_{E_j}, we use e. In most cases the generator will be determined uniquely by the starting representation and the indices used; the only exceptions are the generators of $\operatorname{Hom}(I_{H_j}, I_{E_i})$ and $\operatorname{Hom}(I_{H_j}, I_{P_i})$, where we add the range as a superscript.

form $(\dots 0, g, 0 \dots) \in \prod_{p\in\mathbb{Z}} \mathrm{Hom}(I^p, I^{p+m})$. It should always be clear from context which morphism is meant.

The following table gives a basis of $\mathcal{E}$, and the image of the basis elements under the differential $d_{\mathcal{E}}$.

m	basis of $\mathcal{E}^m$		image under $d_{\mathcal{E}}$
0	h_1	$\mapsto$	$h^e_{11} + h^e_{12}$
	h_2	$\mapsto$	$-h^e_{21} - h^e_{22}$
	e_1	$\mapsto$	$(h^e_{21} - h^e_{11}) + (e_{11} - e_{12})$
	e_2	$\mapsto$	$(h^e_{22} - h^e_{12}) + (e_{22} - e_{21})$
	p_1	$\mapsto$	$e_{21} - e_{11}$
	p_2	$\mapsto$	$e_{12} - e_{22}$
1	h^e_{11}	$\mapsto$	$h^p_{11} - h^p_{12}$
	h^e_{12}	$\mapsto$	$h^p_{12} - h^p_{11}$
	h^e_{21}	$\mapsto$	$h^p_{21} - h^p_{22}$
	h^e_{22}	$\mapsto$	$h^p_{22} - h^p_{21}$
	e_{11}	$\mapsto$	$h^p_{11} - h^p_{21}$
	e_{12}	$\mapsto$	$h^p_{12} - h^p_{22}$
	e_{21}	$\mapsto$	$h^p_{11} - h^p_{21}$
	e_{22}	$\mapsto$	$h^p_{12} - h^p_{22}$
2	h^p_{11}	$\mapsto$	0
	h^p_{12}	$\mapsto$	0
	h^p_{21}	$\mapsto$	0
	h^p_{22}	$\mapsto$	0

For $m \neq 0, 1, 2$, we have $\mathcal{E}^m = 0$. Next, we need to calculate the cohomology of $\mathcal{E}$; please note that we are not going to differentiate notationally between an element of the kernel of $d_{\mathcal{E}}$ and the corresponding element in the cohomology; it should always be clear from context what is meant.

It is easy to see that 1 yields a basis of $\ker d^0_{\mathcal{E}}$ and hence a basis of $H^0\mathcal{E}$. A basis of $\ker d^1_{\mathcal{E}}$ is given by

$$\{d^0_{\mathcal{E}}(f) | f \text{ passes through the above basis of } \mathcal{E}^0 \text{ except for } h_2\}$$

which also generates $\mathrm{im}\, d^0_{\mathcal{E}}$, hence $H^1\mathcal{E}$ vanishes. Finally, a basis of $\mathrm{im}\, d^2_{\mathcal{E}}$ is given by

$$\{d^1_{\mathcal{E}}(h^e_{11}), d^1_{\mathcal{E}}(h^e_{21}), d^1_{\mathcal{E}}(e_{11})\},$$

while a basis of $\ker d^2_{\mathcal{E}}$ is given by our basis of $\mathcal{E}^2$. Accordingly, $H^2\mathcal{E}$ is free of rank 1 and is generated by h^p_{11}.

It is easy to see that the canonical inclusion $\ker d^0 \hookrightarrow \mathcal{E}^0$, and the morphism $H^2\mathcal{E} \to \mathcal{E}^2$ that maps the cohomology class of h^p_{11} to its representant h^p_{11}, combine to a quasi-isomorphism of dg algebras $H\mathcal{E} \to \mathcal{E}$, and we are done. □

5.2. **The 2-sphere and a point.** Next, we would like to discuss the 2-sphere stratified in a point pt and its complement. We had denoted this stratification by $\mathfrak{S}_1$. The corresponding generators of $D^b_{c,\mathfrak{S}_1}(S^2)$ are the skyscraper at pt and the constant sheaf on the sphere, shifted by 1.

Hence we now need to choose an injective resolution I of $\mathcal{W}_{pt} \oplus R_{S^2}[1]$, where $\mathcal{W}_{pt}$ denotes the skyscraper at pt. As previously, by (8) we then have an equivalence

$D^b_{c,\mathfrak{S}_1}(S^2) \simeq \mathcal{D}^f_{\operatorname{End} I}$. Hence the following theorem, which shows that $\operatorname{End} I$ is formal, concludes the proof of Theorem 3 (i) from the introduction.

Theorem 27. *Denote by pt the point in the stratification $\mathfrak{S}_1$ and let $\mathcal{W}_{pt}$ be the skyscraper at this point. For any injective resolution I of $\mathcal{W}_{pt} \oplus R_{S^2}[1]$ the corresponding endomorphism-dg-algebra* $\operatorname{End} I$ *is formal.*

Proof. The proof of this statement will be similar to the proof of Proposition 22. Again we need to acyclify the stratification $\mathfrak{S}_1$. For this we take the acyclic stratification $\mathfrak{A}_2$, where one of the points of $\mathfrak{A}_2$ is taken to be *pt*. Under the equivalence

$$\operatorname{Rep}(Q_{\mathfrak{A}_2}, \rho_{\mathfrak{A}_2}) \xrightarrow{\sim} \operatorname{Sh}(\mathfrak{A}_2) \xrightarrow{\sim} \operatorname{Sh}_{w,\,c,\,\mathfrak{A}_2}(S^2)$$

the representations corresponding to the skyscraper and the constant sheaf on S^2 are

$$W := \begin{tikzcd} 0 & 0 \\ 0 \arrow[u] \arrow[ur] & 0 \arrow[u] \arrow[ul] \\ R \arrow[u] \arrow[ur] & 0 \arrow[u] \arrow[ul] \end{tikzcd} \qquad\qquad C := \begin{tikzcd} R & R \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \\ R \arrow[u] \arrow[ur] & R \arrow[u] \arrow[ul] \end{tikzcd}$$

As in section 5.1, we need to find a sufficiently acyclic resolution of $W \oplus C[1]$; we can do this using the same objects as in (25), and get a resolution

$$I_{H_1} \oplus I_{H_2} \xrightarrow{\begin{pmatrix} h^e_{11} & -h^e_{21} \\ h^e_{12} & -h^e_{22} \\ 0 & 0 \end{pmatrix}} I_{E_1} \oplus I_{E_2} \oplus I_{P_1} \xrightarrow{\begin{pmatrix} e_{11} & -e_{21} & 0 \\ -e_{12} & e_{22} & 0 \end{pmatrix}} I_{P_1} \oplus I_{P_2}$$

Let us denote this resolution by J. By Lemma 24 it is sufficiently acyclic in the sense of Proposition 9, and an argument analogous to Lemma 23 shows that $\operatorname{End} J$ is quasi-equivalent to $\operatorname{End} I$. The following proposition concludes our proof. $\square$

Proposition 28. *For J the resolution of $W \oplus C[1]$ introduced in the previous proof, we have that the corresponding endomorphism algebra* $\operatorname{End} J$ *is formal.*

Proof. We abbreviate $\operatorname{End} J$ to $\mathcal{E}$ and will use the notations from the proof of Proposition 26 to specify a basis of $\mathcal{E}$. Whenever there is need to carry the p-degree of a morphism

$$(\dots 0, f, 0 \dots) \in \prod_{p \in \mathbb{Z}} \operatorname{Hom}(I^p, I^{p+m})$$

along in the notation, we will write $f^{(p)}$ instead of just f.

m	basis of $\mathcal{E}^m$		image under $d_{\mathcal{E}}$
-1	p_1	$\mapsto$	$e_{11} - e_{21}$

m	basis of $\mathcal{E}^m$		image under $d_{\mathcal{E}}$
0	h_1	$\mapsto$	$h^e_{11} + h^e_{12}$
	h_2	$\mapsto$	$-h^e_{21} - h^e_{22}$
	e_1	$\mapsto$	$(h^e_{21} - h^e_{11}) + (e_{11} - e_{12})$
	e_2	$\mapsto$	$(h^e_{22} - h^e_{12}) + (e_{22} - e_{21})$
	$p_1^{(0)}$	$\mapsto$	0
	$p_1^{(1)}$	$\mapsto$	$e_{21} - e_{11}$
	p_2	$\mapsto$	$e_{12} - e_{22}$
	e_{11}	$\mapsto$	$h^p_{21} - h^p_{11}$
	e_{21}	$\mapsto$	$h^p_{21} - h^p_{11}$
1	h^e_{11}	$\mapsto$	$h^p_{11} - h^p_{12}$
	h^e_{12}	$\mapsto$	$h^p_{12} - h^p_{11}$
	h^e_{21}	$\mapsto$	$h^p_{21} - h^p_{22}$
	h^e_{22}	$\mapsto$	$h^p_{22} - h^p_{21}$
	h^p_{11}	$\mapsto$	0
	h^p_{21}	$\mapsto$	0
	p_1	$\mapsto$	0
	e_{11}	$\mapsto$	$h^p_{11} - h^p_{21}$
	e_{12}	$\mapsto$	$h^p_{12} - h^p_{22}$
	e_{21}	$\mapsto$	$h^p_{11} - h^p_{21}$
	e_{22}	$\mapsto$	$h^p_{12} - h^p_{22}$
2	h^p_{11}	$\mapsto$	0
	h^p_{12}	$\mapsto$	0
	h^p_{21}	$\mapsto$	0
	h^p_{22}	$\mapsto$	0

For all other m we have $\mathcal{E}^m = 0$. Next, we need to calculate bases of the kernel and image of the differential $d_{\mathcal{E}}$:

	basis
$\ker d_{\mathcal{E}}^{-1}$	$\emptyset$
$\operatorname{im} d_{\mathcal{E}}^{-1}$	$\{e_{11} - e_{21}\}$
$\ker d_{\mathcal{E}}^0$	$\{1, e_{11} - e_{21}, p_1^{(0)}\}$
$\operatorname{im} d_{\mathcal{E}}^0$	$\{d^0_{\mathcal{E}}(h_1), d^0_{\mathcal{E}}(e_1), d^0_{\mathcal{E}}(e_2), d^0_{\mathcal{E}}(p_1^{(1)}), d^0_{\mathcal{E}}(p_2), d^0_{\mathcal{E}}(e_{11})\}$
$\ker d_{\mathcal{E}}^1$	$\{d^0_{\mathcal{E}}(h_1), d^0_{\mathcal{E}}(e_1), d^0_{\mathcal{E}}(e_2), d^0_{\mathcal{E}}(p_1^{(1)}), d^0_{\mathcal{E}}(p_2), h^p_{11}, h^p_{21}, p_1\}$
$\operatorname{im} d_{\mathcal{E}}^1$	$\{d^1_{\mathcal{E}}(h^e_{11}), d^1_{\mathcal{E}}(h^e_{21}), d^1_{\mathcal{E}}(e_{11})\}$
$\ker d_{\mathcal{E}}^2$	$\{h^p_{11}, h^p_{12}, h^p_{21}, h^p_{22}\}$

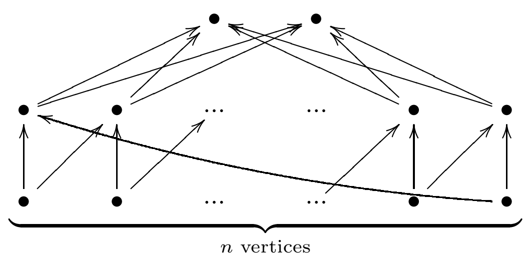
Thus the -1^{st} cohomology of $\mathcal{E}$ vanishes. $H^0\mathcal{E}$ is free of rank 2, with basis $\{1, p_1^{(0)}\}$, $H^1\mathcal{E}$ is free of rank 2 as well, with basis $\{h^p_{11}, p_1\}$, while $H^2\mathcal{E}$ is free of rank 1 with basis $\{h^p_{11}\}$. Now consider the morphism $H\mathcal{E} \to \mathcal{E}$ that maps every homology class to its representative that we just specified. Since our system of representatives is closed under multiplication, this is a morphism of dg algebras and is a quasi-isomorphism by construction. $\square$

5.3. **The 2-sphere and n points.** Finally, we would like to consider for $n \geq 2$ the stratification $\mathfrak{S}_n$, which decomposes the 2-sphere into n points and their complement. Since the big stratum is no longer simply connected, the skyscrapers at the

n points and the constant sheaf generate only a subcategory of the corresponding constructible derived category $D^b_{c,\mathfrak{S}_n}(S^2)$. However, since our focus is on formality in general, we will show that the corresponding dg algebra is formal as well. Our final result is the proof of Theorem 4 from the introduction:

Theorem 29. *For $n \geq 2$, let $\mathcal{W}_1, \ldots, \mathcal{W}_n$ be the skyscrapers at the n points of the stratification $\mathfrak{S}_n$. Then for any injective resolution I of $\mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n \oplus R_{S^2}[1]$, the dg algebra* $\operatorname{End} I$ *is formal.*

Proof. We consider the acyclic stratification $\mathfrak{A}_n$ corresponding to $\mathfrak{S}_n$, which consists of n points $P_1, \ldots, P_n$ (for simplicity we assume they are on the equator), the equator pieces in between, $E_1, \ldots, E_n$, and the two hemi-spheres H_1, H_2. The corresponding quiver is given by



where we identify by relations all paths having the same starting and end point. Under the equivalence

$$\operatorname{Rep}(Q_{\mathfrak{A}_n}, \rho_{\mathfrak{A}_n}) \xrightarrow{\sim} \operatorname{Sh}(\mathfrak{A}_n) \xrightarrow{\sim} \operatorname{Sh}_{w,\,c,\,\mathfrak{A}_n}(S^2)$$

the constant sheaf R_{S^2} corresponds to the representation C that has R at every vertex and the identity at every arrow. The skyscraper $\mathcal{W}_i$ corresponds to the representation that has R at the ith vertex in the lower row and zeros everywhere else. We denote this object by W_i.

Next, we need sufficiently acyclic objects. As before, we take for every stratum S of $\mathfrak{A}_n$ the representation corresponding to the constant sheaf along the closure of S. More precisely, let $i : \overline{S} \hookrightarrow S^2$ be the inclusion, and denote by I_S the image of $i_! R_{\overline{S}}$ under the equivalence $\operatorname{Sh}_{w,c,\mathfrak{A}_n} \xrightarrow{\sim} \operatorname{Rep}(Q_{\mathfrak{A}_n}, \rho_{\mathfrak{A}_n})$. This yields the following representation: At the vertex corresponding to S we have R, as well as at every vertex from which we have a path to the vertex S. The other vertices get a zero. Finally we assign to an arrow the identity map, if possible, and else the zero morphism. As in Lemma 24 we see that those objects do not have higher extensions.

Again, we need to study the morphism spaces between the objects I_S. They vanish in all except for the following cases, in which they are free of rank 1:

$\operatorname{Hom}(I_{H_j}, I_{H_j})$	for $j = 1$ or 2;
$\operatorname{Hom}(I_{H_j}, I_{E_i})$	for $i = 1, \ldots, n$ und $j = 1, 2$;
$\operatorname{Hom}(I_{H_j}, I_{P_i})$,	for $i = 1, \ldots, n$ und $j = 1, 2$;
$\operatorname{Hom}(I_{E_i}, I_{E_i})$	for $i = 1, \ldots, n$;
$\operatorname{Hom}(I_{E_i}, I_{P_i})$	for $i = 1, \ldots, n$;
$\operatorname{Hom}(I_{E_1}, I_{P_n})$ and $\operatorname{Hom}(I_{E_i}, I_{P_{i-1}})$	for $i = 2, \ldots, n$;
$\operatorname{Hom}(I_{P_i}, I_{P_i})$	for $i = 1, \ldots, n$.

Each of these morphism spaces is generated by a canonical morphism, the one that is given by the identity on R wherever possible and zeros everywhere else. We are going to use the same notations for those morphisms as before: h_j, e_i and p_i are meant to be the generators of $\mathrm{Hom}(I_{H_j}, I_{H_j})$, $\mathrm{Hom}(I_{E_i}, I_{E_i})$ and $\mathrm{Hom}(I_{P_i}, I_{P_i})$, respectively. The generator of $\mathrm{Hom}(I_{H_j}, I_{E_i})$ is denoted by h^e_{ji}, that of $\mathrm{Hom}(I_{H_j}, I_{P_i})$ by h^p_{ji} and that of $\mathrm{Hom}(I_{E_i}, I_{P_k})$ by e_{ik}. For notational simplicity we take indices modulo n, i.e. for $i = 1$, we get $e_{i(i-1)} = e_{1n}$, etc.

We now need to find an acyclic resolution of $W_1 \oplus \cdots \oplus W_n \oplus C$. With the above notations, we get the following resolution:

$$I_{H_1} \oplus I_{H_2} \xrightarrow{\begin{pmatrix}\partial^0\\ 0\end{pmatrix}} \begin{matrix} I_{E_1} \oplus \ldots \oplus I_{E_n} \\ \oplus \\ I_{P_1} \oplus \ldots \oplus I_{P_n} \end{matrix} \xrightarrow{\begin{pmatrix}\partial^1 & 0\end{pmatrix}} I_{P_1} \oplus \ldots \oplus I_{P_n}$$

where ∂^0 is given by the matrix

$$\begin{pmatrix} h^e_{11} & -h^e_{21} \\ \vdots & \vdots \\ h^e_{1n} & -h^e_{2n} \end{pmatrix}$$

and ∂^1 by

$$\begin{pmatrix} e_{11} & -e_{21} & 0 & \cdots & & 0 \\ 0 & e_{22} & 0 & \cdots & & 0 \\ \vdots & 0 & \ddots & & & \vdots \\ 0 & \vdots & \cdots & & 0 & -e_{n(n-1)} \\ -e_{1n} & 0 & \cdots & & 0 & e_{nn} \end{pmatrix}$$

Using an argument similar to Lemma 23, the result follows by the next proposition. □

Proposition 30. *For J the resolution of $W_1 \oplus \cdots \oplus W_n \oplus C$ introduced in the previous proof, the corresponding endomorphism-dg-algebra* $\mathrm{End}\, J$ *is formal.*

Proof. This proof is going to be a little more complex than those of the analogous statements before. We are going to find a dg-sub-algebra $\mathcal{U}$ of $\mathrm{End}\, J$ and a two-sided ideal $\mathcal{I}$ of $\mathcal{U}$, that yield a sequence of quasi-isomorphisms

$$\mathrm{End}\, J \hookleftarrow \mathcal{U} \twoheadrightarrow \mathcal{U}/I \hookleftarrow H(\mathcal{U}/\mathcal{I})$$

For brevity, we write $\mathcal{E}$ instead of $\mathrm{End}\, J$. Using the same notations as before we would like to give a basis of the free module $\mathcal{E}$:

m	rank of $\mathcal{E}^m$	basis of $\mathcal{E}^m$		image under $d_{\mathcal{E}}$	
-1	n	p_i	$\mapsto$	$e_{ii} - e_{(i+1)i}$	$i = 1, \ldots, n$

m	rank of $\mathcal{E}^m$	basis of $\mathcal{E}^m$		image under $d_{\mathcal{E}}$	
0	$5n+2$	h_1	$\mapsto$	$\sum_{i=1}^{n} h^e_{1i}$	
		h_2	$\mapsto$	$-\sum_{i=1}^{n} h^e_{2i}$	
		e_i	$\mapsto$	$(h^e_{2i} - h^e_{1i}) + (e_{ii} - e_{i(i-1)})$	$i = 1, \dots, n$
		$p^{(0)}_i$	$\mapsto$	0	$i = 1, \dots, n$
		$p^{(1)}_i$	$\mapsto$	$e_{(i+1)i} - e_{ii},$	$i = 1, \dots, n$
		e_{ii}	$\mapsto$	$h^p_{2i} - h^p_{1i}$	$i = 1, \dots, n$
		$e_{(i+1)i}$	$\mapsto$	$h^p_{2i} - h^p_{1i}$	$i = 1, \dots, n$
1	$7n$	h^e_{1i}	$\mapsto$	$h^p_{1i} - h^p_{1(i-1)}$	$i = 1, \dots, n$
		h^e_{2i}	$\mapsto$	$h^p_{2i} - h^p_{2(i-1)}$	$i = 1, \dots, n$
		h^p_{ji}	$\mapsto$	0	$j = 1, 2,\ i = 1, \dots, n$
		e_{ii}	$\mapsto$	$h^p_{1i} - h^p_{2i}$	$i = 1, \dots, n$
		$e_{(i+1)i}$	$\mapsto$	$h^p_{1i} - h^p_{2i}$	$i = 1, \dots, n$
		p_i	$\mapsto$	0	$i = 1, \dots, n$
2	$2n$	h^p_{ji}	$\mapsto$	0	$j = 1, 2,\ i = 1, \dots, n$

Next, we need to specify a basis of the kernel and image of the differential $d_{\mathcal{E}}$:

	basis
$\ker d^{-1}_{\mathcal{E}}$	$\emptyset$
$\operatorname{im} d^{-1}_{\mathcal{E}}$	the basis of $\mathcal{E}^{-1}$ as in the table above
$\ker d^0_{\mathcal{E}}$	$\{1, p^{(0)}_i, (e_{ii} - e_{(i+1)i}) \mid i = 1, \dots, n\}$
$\operatorname{im} d^0_{\mathcal{E}}$	$\{d^0_{\mathcal{E}}(h_1), d^0_{\mathcal{E}}(e_i), d^0_{\mathcal{E}}(p^{(1)}_i), d^0_{\mathcal{E}}(e_{ii}) \mid i = 1, \dots, n\}$
$\ker d^1_{\mathcal{E}}$	$\{d^0_{\mathcal{E}}(h_1), d^0_{\mathcal{E}}(e_i), d^0_{\mathcal{E}}(p^{(1)}_i), h^p_{ji}, p_i \mid j = 1, 2, i = 1, \dots, n\}$
$\operatorname{im} d^1_{\mathcal{E}}$	$\{d^1_{\mathcal{E}}(h^e_{1i}), d^1_{\mathcal{E}}(h^e_{2i}), d^1_{\mathcal{E}}(e_{11}) \mid i = 2, \dots, n\}$
$\ker d^2_{\mathcal{E}}$	the basis of $\mathcal{E}^2$ as in the table above

From this table we can read the cohomology of $\mathcal{E}$:

$$H^{-1}\mathcal{E} = 0, \quad H^0\mathcal{E} = \langle 1,\ p^{(0)}_i \rangle_{i=1,\dots,n}, \quad H^1\mathcal{E} = \langle h^p_{1i}, p_i \rangle_{i=1,\dots,n} \quad \text{and } H^2\mathcal{E} = \langle h^p_{11} \rangle.$$

Unfortunately, unlike before, we cannot find a quasi-isomorphism $H\mathcal{E} \to \mathcal{E}$. This is because in this situation, there is no system of representants of the generators of $H\mathcal{E}$ that form a sub-algebra of $\mathcal{E}$. The problem is that when multiplying the representants of $H^1\mathcal{E}$, we get all of the h^p_{1i}. Hence the first step is to identify the h^p_{1i}, which can be done by defining a subalgebra $\mathcal{U}$ of $\mathcal{E}$ and dividing it by a suitable ideal $\mathcal{I}$.

Consider the dg algbra $\mathcal{U}$ given by the following basis:

m	rank of $\mathcal{U}^m$	basis of $\mathcal{U}^m$	image under $d_{\mathcal{U}}$
-1	0	$\emptyset$	

m	rank of $\mathcal{U}^m$	basis of $\mathcal{U}^m$		image under $d_{\mathcal{U}}$	
0	$2n+3$	h_1	$\mapsto$	$\sum_{i=1}^{n} h^e_{1i}$	
		h_2	$\mapsto$	$-\sum_{i=1}^{n} h^e_{2i}$	
		e_i	$\mapsto$	$(h^e_{2i} - h^e_{1i}) + (e_{ii} - e_{i(i-1)})$	$i = 1, \ldots, n$
		$p_i^{(0)}$	$\mapsto$	0	$i = 1, \ldots, n$
		$\sum_{i=1}^n p_i^{(1)}$	$\mapsto$	$\sum_{i=1}^n (e_{(i+1)i} - e_{ii})$	
1	$5n+1$	h^e_{1i}	$\mapsto$	$h^p_{1i} - h^p_{1(i-1)}$	$i = 1, \ldots, n$
		h^e_{2i}	$\mapsto$	$h^p_{2i} - h^p_{2(i-1)}$	$i = 1, \ldots, n$
		h^p_{1i}	$\mapsto$	0	$i = 1, \ldots, n$
		e_{11}	$\mapsto$	$h^p_{11} - h^p_{21}$	
		e_{1n}	$\mapsto$	$h^p_{1n} - h^p_{2n}$	
		$e_{ii} - e_{i(i-1)}$	$\mapsto$	$h^p_{1i} - h^p_{2i} - (h^p_{1(i-1)} - h^p_{2(i-1)})$	$i = 2, \ldots, n$
		p_i	$\mapsto$	0	$i = 1, \ldots, n$
2	$2n$	h^p_{ji}	$\mapsto$	0	$j = 1, 2,\ i = 1, \ldots, n$

It is easy to see that $\mathcal{U}$ is closed by multiplication, and that the inclusion $\mathcal{U} \hookrightarrow \mathcal{E}$ is a quasi-isomorphism.

Consider the two-sided ideal $\mathcal{I}$ of $\mathcal{U}$ that is given by $\mathcal{I}^0 = 0$, $\mathcal{I}^1 = \langle h^e_{1i} \rangle_{i=2,\ldots,n}$ and $\mathcal{I}^2 = \langle h^p_{1i} - h^p_{1(i-1)} \rangle_{i=2,\ldots,n}$. The cohomology of $\mathcal{I}$ vanishes, hence by the long exact cohomology sequence the projection $\mathcal{U} \twoheadrightarrow \mathcal{U}/\mathcal{I}$ is a quasi-isomorphism.

Now in $\mathcal{U}/\mathcal{I}$ we have the following relations:

$$h^p_{11} = h^p_{1i} \text{ for } i = 2, \ldots, n$$

Hence the following system of generators of the cohomology of $\mathcal{U}/\mathcal{I}$ is closed under multiplication:

$$H^0(\mathcal{U}/\mathcal{I}) = \langle 1, p_i^{(0)} \rangle_{i=1,\ldots,n},\ H^1(\mathcal{U}/\mathcal{I}) = \langle h^p_{1i}, p_i \rangle_{i=1,\ldots,n} \text{ und } H^2(\mathcal{U}/\mathcal{I}) = \langle h^p_{11} \rangle$$

Mapping each of those generators to the corresponding representant yields the final quasi-isomorphism in the chain

$$\mathcal{E} \hookleftarrow \mathcal{U} \twoheadrightarrow \mathcal{U}/I \hookleftarrow H(\mathcal{U}/\mathcal{I})$$

and we are done. □

References

[Bal06] Anne Balthasar, *Formalität der konstruierbaren derivierten Kategorie am Beispiel der Sphären.* Diplomarbeit, Freiburg, 2006, available online at `http://www.freidok.uni-freiburg.de/volltexte/2540/`

[BL94] Joseph Bernstein and Valery Lunts, *Equivariant sheaves and functors.* Lecture Notes in Mathematics, vol. 1578, Springer-Verlag, Berlin, 1994

[DGMS75] Pierre Deligne, Phillip Griffith, John Morgan, and Dennis Sullivan, *Real homotopy theory of Kähler manifolds.* Invent. Math. **29** (1975), no. 3, 245–274

[GR97] Peter Gabriel and Andrei V. Roiter. *Representations of finite-dimensional algebras.* Springer-Verlag, Berlin, 1997. Translated from the Russian, With a chapter by B. Keller, Reprint of the 1992 English translation.

[Gui05] Stéphane Guillermou, *Equivariant derived category of a complete symmetric variety.* Represent. Theory **9** (2005), 526–577

[Ive86] Birger Iversen, *Cohomology of sheaves.* Universitext, Springer-Verlag, Berlin, 1986

[KS94] Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1994. With a chapter in French by Christian Houzel. Corrected reprint of the 1990 original

[Kel94] Bernhard Keller, *Deriving DG categories.* Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102

[Lun95] Valery Lunts, *Equivariant sheaves on toric varieties.* Compositio Math. **96** (1995), no. 1, 63–83, 1995

[Sch08] Olaf M. Schnürer, *Equivariant sheaves on flag varieties, DG modules and formality.* PhD thesis, Freiburg, to appear

[Soe01] Wolfgang Soergel, *Langlands' philosophy and Koszul duality.* Algebra–representation theory (Constanta, 2000), NATO Sci. Ser. II Math. Phys. Chem. **28**, Kluwer, 2001, 379–414

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